Algebraic Number Theory Final Exam

March 3 2015

Please do not cheat. Good Luck! This exam is of 40 marks. There are 4 questions. (40)

1. Minkowski's bound implies that in every ideal class [I] there is an any ideal $J \in [I]$ with

$$N(J) \leqslant \left(\frac{4}{\pi}\right)^s \cdot \frac{n!}{n^n}\sqrt{|\Delta_K|}.$$

where Δ_K is the discriminant and $n = r + 2s = [K : \mathbb{Q}]$. Use it to compute the class number of
a. $\mathbb{Q}(\sqrt{10})$. (5)
b. $\mathbb{Q}(\sqrt{-163})$ (5)

2 Let $K = \mathbb{Q}(\sqrt{-5})$ and $L = K(\sqrt{5})$. Show that no prime of K ramifies in L. You may use that

$$\mathcal{O}_L = \mathbb{Z}[i, \frac{1+i\sqrt{5}}{2}]$$

(8)

3. Factorize $< 2 >$ in $\mathbb{Q}(\zeta_5)$, where $\zeta_5 = \exp(2\pi i/5)$. (6)

4. Let $K = \mathbb{Q}(\theta)$ with θ in $\mathcal{O}_K$. Let $n = [K : \mathbb{Q}]$. Show that if $\Delta(1, \theta, \theta^2, \ldots, \theta^{n-1})$ is square-free then $\mathcal{O}_K = \mathbb{Z}[\theta]$. (8)

5. Let R be an integral domain and K its quotient field. Let M be an R-submodule of a finite dimensional K-vector space. Show

$$M = \bigcap_{p \ maximal} M_p$$

where $M_p = R_p M$. (8).